Worksheet 3.

In this worksheet, we will explore simplifying functions and algebraic expressions involving rational functions, difference quotient. We will also discuss simplifying expressions involving square roots.

Examples: $f(x) = \dfrac{x^2-4}{x^2-3x+2}$, $f(t) = \dfrac{1}{t} - \dfrac{1}{t^2+t}$, $f(x) = \dfrac{x-3}{\sqrt{x+6}-3}$

Identities

- $a^2 - b^2 = (a-b)(a+b)$ *difference between squares.*
- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ *difference between cubes.*
- $a^3 + b^3 = (a+b)(a^2-ab+b^2)$ *sum between cubes.*
- $(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2$ *perfect square binomial.*
- $(a-b)^2 = (a-b)(a-b) = a^2-2ab+b^2$ *perfect square binomial.*
- $(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2) = a^3+3a^2b+3ab^2+b^3$
- $(a-b)^3 = (a-b)(a-b)^2 = (a-b)(a^2-2ab+b^2) = a^3-3a^2b+3ab^2-b^3$

Simplifying Rational Functions and Difference Quotient .

Simplifying rational functions often requires us to cancel common factors between numerator and the denominator.

Examples:

1. Simplify the rational function $f(t) = \dfrac{x^4-4x^2}{x^3-6x^2+8x}$.
   In order to simplify, we factor out both numerator and the denominator.
   $f(x) = \dfrac{x^2(x^2-4)}{x(x^2-6x+8)} = \dfrac{x(x-2)(x+2)}{(x-2)(x-4)} = \dfrac{x(x+2)}{x-4}$

2. If $f(t) = t^2 - 15t - 8$ and $h \neq 0$, then simplify $\dfrac{f(t+h)-f(t)}{h}$.
   $\dfrac{f(t+h)-f(t)}{h} = \dfrac{[(t+h)^2 - 15(t+h) - 8] - [t^2 - 15t - 8]}{h}$
   We use the *perfect square binomial expansion*, $(a+b)^2 = a^2 + 2ab + b^2$ to expand $(t+h)^2$.
   $\dfrac{f(t+h)-f(t)}{h} = \dfrac{t^2 + 2ht + h^2 - 15t - 15h - 8 - t^2 + 15t + 8}{h}$
   We can now simplify the terms in the numerator.
   expand$(t+h)^2$.
   $\dfrac{f(t+h)-f(t)}{h} = \dfrac{2ht + h^2 - 15h}{h}$ Factor out the common factor h from the numerator.
   $= \dfrac{h(2t+h-15)}{h}$ Now we can cancel h.
   $= 2t + h - 15$

   Sometimes, simplifying rational functions requires us to first combine terms by taking the common denominator.

3. Simplify the rational function $f(x) = \dfrac{4}{x+1} - \dfrac{3}{x+2}$.
   In order to simplify, we first need to take the common denominator. Note that the common denominator is $(x+1)(x+2)$

$$\begin{aligned} f(x) &= \frac{4(x+2)}{(x+1)(x+2)} - \frac{3(x+1)}{(x+1)(x+2)} \\ &= \frac{4(x+2) - 3(x+1)}{(x+1)(x+2)} \\ &= \frac{4x+8-3x-3}{(x+1)(x+2)} \\ &= \frac{x+5}{(x+1)(x+2)} \end{aligned}$$

4. If $f(x) = \dfrac{1}{x^2}$ and $h \neq 0$, then simplify $\dfrac{f(x+h)-f(x)}{h}$.
   First, simplify $f(x+h) - f(x)$.

$$\begin{aligned} f(x+h) - f(x) &= \frac{1}{(x+h)^2} - \frac{1}{x^2} && \text{Take the common denominator..} \\ &= \frac{x^2 - (x+h)^2}{x^2(x+h)^2} && \text{Simplify the numerator.} \\ &= \frac{x^2 - (x^2+2hx+h^2)}{x^2(x+h)^2} && \text{Simplify the numerator.} \\ &= \frac{x^2 - x^2 - 2hx - h^2}{x^2(x+h)^2} \\ &= \frac{-2hx - h^2}{x^2(x+h)^2} \end{aligned}$$

   Now, we simplify $\dfrac{f(x+h)-f(x)}{h}$.

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\frac{-2hx-h^2}{x^2(x+h)^2}}{h}. \\ &= \frac{-2hx-h^2}{x^2(x+h)^2} \cdot \frac{1}{h} \\ &= \frac{-2hx-h^2}{hx^2(x+h)^2} && \text{Factor out the common factor } h \text{ from the numerator.} \\ &= \frac{h(-2x-h)}{hx^2(x+h)^2} && \text{Now we can cancel } h. \\ &= \frac{-2x-h}{x^2(x+h)^2}. \end{aligned}$$

Try these:

1. If $f(x) = x^3 - 4x + 7$ and $h \neq 0$, then simplify $\dfrac{f(x+h)-f(x)}{h}$.

2. Simplify $\dfrac{2t^3 - 18t}{2t^3 - 10t^2 + 12t}$.

3. Simplify the rational function $f(x) = \dfrac{4}{x^2-1} - \dfrac{3}{(x-1)(2x+1)}$.

Simplifying Expressions Involving Square Roots

In Calculus, we often require to simplify expressions involving square root terms. Recall how you rationalize irrational denominators.

Examples:

1. $\dfrac{5}{\sqrt{3}} = \dfrac{5}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{5\sqrt{3}}{3}$

2. $\dfrac{3}{\sqrt{5}-\sqrt{2}} = \dfrac{3}{\sqrt{5}-\sqrt{2}} \cdot \dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \dfrac{3(\sqrt{5}+\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2} = \dfrac{3(\sqrt{5}+\sqrt{2})}{25-4} = \dfrac{3(\sqrt{5}+\sqrt{2})}{21}$

3. $\dfrac{2}{\sqrt{4}+\sqrt{3}} = \dfrac{2}{\sqrt{4}+\sqrt{3}} \cdot \dfrac{\sqrt{4}-\sqrt{3}}{\sqrt{4}-\sqrt{3}}$

   $$= \frac{2(\sqrt{4}-\sqrt{3})}{(\sqrt{4})^2-(\sqrt{3})^2} = \frac{2(\sqrt{4}-\sqrt{3})}{16-9} = \frac{2(\sqrt{4}-\sqrt{3})}{7}$$

In example 2 & 3, in order to rationalize the denominator, we multiply by the *conjugate* of the term, which is changing the sign between the two terms in to the opposite sign.
The method is the same with algebraic expressions instead of number, even though it looks complicated.
Examples:

1. To simplify $\dfrac{2x}{\sqrt{x+9}-3}$, we multiply both numerator and the denominator by the conjugate of $(\sqrt{x+9}-3)$, which is $(\sqrt{x+9}+3)$.

   $$\begin{aligned}\frac{2x}{\sqrt{x+9}-3} &= \frac{2x}{\left(\sqrt{x+9}-3\right)} \cdot \frac{\left(\sqrt{x+9}+3\right)}{\left(\sqrt{x+9}+3\right)} \\ &= \frac{2x\left(\sqrt{x+9}+3\right)}{\left(\sqrt{x+9}\right)^2-3^2} \\ &= \frac{2x\left(\sqrt{x+9}+3\right)}{(x+9)-9} \\ &= \frac{2x\left(\sqrt{x+9}+3\right)}{x} \\ &= 2\left(\sqrt{x+9}+3\right)\end{aligned}$$

   Note that, to simplify the product between $\left(\sqrt{x+9}-3\right)$ and its conjugate, we use difference between squares formula, $a^2-b^2=(a-b)(a+b)$ backwards. This is actually the main purpose of multiplying by the conjugate.

2. Note that the function $f(x) = \dfrac{\sqrt{x^2+9}-5}{x+4}$ is not defined when $x=-4$, so the domain is $(-\infty,-4)\cup(-4,\infty)$.
   We can rewrite this function in a different format, so that it is defined at $x=-4$. To simplify, we multiply both numerator and the denominator by the conjugate of numerator.

$$\begin{aligned}
f(x) &= \frac{(\sqrt{x^2+9}-5)}{(x+4)}\,\frac{(\sqrt{x^2+9}+5)}{(\sqrt{x^2+9}+5)} \\
&= \frac{(x^2+9)-25}{(x+4)(\sqrt{x^2+9}+5)} \\
&= \frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)} \\
&= \frac{(x-4)(x+4)}{(x+4)(\sqrt{x^2+9}+5)} \\
&= \frac{x-4}{\sqrt{x^2+9}+5}
\end{aligned}$$

The simplified form of f is defined when $x = -4$. This is very important in Calculus, when we find limits of functions using limit laws.

Try these:

1. Rationalize the denominator.
   $\dfrac{x-3}{\sqrt{x+6}-3}$
2. Rationalize the denominator.
   $\dfrac{x^2+3x}{\sqrt{x+5}-\sqrt{5}}$
3. Rewrite the following function so that it is defined at $t = 5$.
   $\dfrac{\sqrt{3t+10}-5}{t-5}$