Worksheet 5.

In this worksheet, we will review rules of exponents and logarithms. In your classes, you have likely already seen that the rules of exponents are useful for simplifying expressions and applying the power rule for differentiation. The rules of logarithms will be useful for a technique called logarithmic differentiation.

Properties of Exponents Let a and b be real numbers, variables, or algebraic expressions. Let m and n be integers. Then

1. $a^m a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$
4. $a^0 = 1,\ a \neq 0$
5. $(ab)^m = a^m b^m$
6. $(a^m)^n = a^{mn}$
7. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},$

Examples: Use the properties of exponents to rewrite the following expressions.

1. $\sqrt{x}(x^2+3x+1)$

   Solution $\sqrt{x}(x^2+3x+1) = x^{1/2}(x^2+3x+1) = x^{5/2} + 3x^{3/2} + x^{1/2}$

2. $\dfrac{x^5 - x^2}{x^3}$

   Solution $\dfrac{x^5 - x^2}{x^3} = \dfrac{x^5}{x^3} - \dfrac{x^2}{x^3} = x^{5-3} - x^{2-3} = x^2 - x^{-1}$

3. $\left(\dfrac{x-1}{\sqrt{x}}\right)^3$

   Solution $$\left(\frac{x-1}{\sqrt{x}}\right)^3 = \frac{(x+1)^3}{\sqrt{x}^3} = \frac{x^3 - 3x^2 + 3x - 1}{x^{3/2}} = \frac{x^3}{x^{3/2}} - \frac{3x^2}{x^{3/2}} + \frac{3x}{x^{3/2}} - \frac{1}{x^{3/2}}$$
   $$= x^{3-3/2} - 3x^{2-3/2} + 3x^{1-3/2} - x^{-3/2} = x^{3/2} - 3x^{1/2} + 3x^{-1/2} - x^{-3/2}$$

Please note that the answers given above leave negative exponents of x in the numerator since this is the most useful form for applying the power rule for differentiation in calculus.

Try these:

1. $x^3(x^5 - x^2 + 6)$
2. $x\sqrt{x} + \dfrac{1}{x^2\sqrt{x}}$
3. $\dfrac{x^2 - 2\sqrt[3]{x}}{x^4}$

<u>*Link to Calculus:*</u> Differentiate the following functions. Notice that we can avoid using the product or quotient rules in these examples (if desired) if we expand the functions and rewrite using rules of exponents.

1. $y = x^3(x^5 - x^2 + 6)$
2. $y = x\sqrt{x} + \dfrac{1}{x^2\sqrt{x}}$
3. $y = \dfrac{x^2 - 2\sqrt[3]{x}}{x^4}$

Properties of Logarithms Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	**Natural Logarithm**
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v$	$\ln\left(\frac{u}{v}\right) = \ln u + \ln v$
3. Power Property:	$\log_a(u^v) = v\log_a u$	$\ln(u^v) = v\ln u$

<u>*Examples*</u>:

1. Use the properties of logarithms to rewrite and simplify the logarithmic expression.
   (a) $\log_2(4^2 \cdot 3^4)$

   Solution $\log_2(4^2 \cdot 3^4) = \log_2(4^2) + \log_2(3^4) = 2\log_2 4 + 4\log_2 3 = 4 + 4\log_2 3$

   (b) $\ln(5e^6)$

   Solution $\ln(5e^6) = \ln 5 + \ln e^6 = \ln 5 + 6\ln e = 6 + \ln 5.$

   (c) $\log_3 81^{-3}$

   Solution $\log_3 81^{-3} = -3\log_3(81) = -3 \cdot 4 = -12$

2. Condense the expression to the logarithm of a single quantity.
   (a) $4[\ln z + \ln(z+5)] - 2\ln(z-5)$

   Solution
   $$\begin{aligned} 4[\ln z + \ln(z+5)] - 2\ln(z-5) &= 4[\ln z(z+5)] - 2\ln(z-5) \\ &= \ln[z(z+5)]^4 - \ln(z-5)^2 = \ln[z^4(z+5)^4] - \ln(z-5)^2 \\ &= \ln\left(\frac{z^4(z+5)^4}{(z-5)^2}\right) \end{aligned}$$

   (b) $3\log_3 x + 4\log_3 y - 4\log_3 z$

   Solution $3\log_3 x + 4\log_3 y - 4\log_3 z = \log_3 x^3 + \log_3 y^4 - \log_3 z^4 = \log_3\left(\dfrac{x^3y^4}{z^4}\right)$

3. Use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms.

   (a) $\log 4x^2y$

   Solution $\log 4x^2y = \log 4 + \log(x^2) + \log y = \log 4 + 2\log x + \log y$

   (b) $\log_3 \frac{xy^4}{z^5}$

   Solution $\log_3 \frac{xy^4}{z^5} = \log_3(xy^4) - \log_3(z^5) = \log_3 x + \log(y^4) - \log_3(z^5) = \log_3 x + 4\log y - 5\log_3 z$

Try these:

1. Rewrite and simplify as much as possible $\ln \dfrac{1}{\sqrt{e}} - \ln \sqrt[4]{e^5} + 2\ln e^3$
2. Condense $2\ln 8 + 5\ln(z-4)$
3. Expand $\ln \sqrt[4]{x^3(x^2+3)}$
4. Expand $\ln \frac{x^4(x+1)^2}{(x^2-4)(x+3)^2}$
5. Expand $\ln(x^{x^2})$
6. Expand $\ln(\sin x)^{\cos x}$

Link to Calculus: Once you have covered logarithmic differentiation in your calculus course, use that technique to differentiate the following functions.

1. $y = \sqrt[4]{x^3(x^2+3)}$
2. $y = \frac{x^4(x+1)^2}{(x^2-4)(x+3)^2}$
3. $y = x^{x^2}$
4. $y = (\sin x)^{\cos x}$

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an exponential equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a logarithmic equation in exponential form and apply the Inverse Property of exponential functions.

	One-to-One Property	Inverse Property
Exponential Functions	$a^u = a^v \iff u = v$	$a^{\log_a x} = x$
Logarithmic Functions	$\log_a u = \log_a v \iff u = v$	$\log_a(a^x) = x$

Examples Solve for x.

1. $8^{x^2+3} = 8^{2x+2}$

   Solution By the one-to-one property, $x^2 + 3 = 2x + 2$ so $x^2 - 2x + 1 = (x - 1)^2 = 0$. Thus $x = 1$.

2. $2 \ln \sqrt{x - 8} = 10$

   Solution Recognizing that the square root can be written as the rational exponent $\frac{1}{2}$, we can rewrite the equation as
   $$2 \ln(x - 8)^{\frac{1}{2}} = 10.$$
   Using the power property, we can rewrite this as
   $$2\left(\frac{1}{2}\ln(x-8)\right) = \ln(x - 8) = 10.$$
   Then
   $$e^{\ln(x-8)} = e^{10}.$$
   By the inverse property this gives
   $$x - 8 = e^{10}.$$
   Thus $x = 8 + e^{10}$.

3. $\ln(x + 1) - \ln(x - 1) = \ln x$

   Solution Begin by condensing the left-hand side of the equation:
   $$\ln\left(\frac{x+1}{x-1}\right) = \ln x.$$
   Then by the one-to-one property, we have
   $$\frac{x+1}{x-1} = x \Rightarrow x + 1 = x(x - 1) \Rightarrow x + 1 = x^2 - x \Rightarrow 0 = x^2 - 2x - 1$$
   $$\Rightarrow x = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2(1)} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$
   Notice, however, that neither $\ln x$ nor $\ln(x - 1)$ are defined when $x = 1 - \sqrt{2}$, so the only solution is $x = 1 + \sqrt{2}$.

Try these: Solve for x.

1. $6(2^{3x-1}) - 7 = 9$
2. $\ln x + \ln(x + 1) = 1$
3. $4x^2e^{3x} + 8xe^{3x} = 0$

Link to Calculus: For what values of x is the *derivative* of the function equal to 0?

1. $f(x) = e^{-x} - e^{-2x}$
2. $f(x) = x \ln x$
3. $f(x) = \dfrac{(\ln x)}{\sqrt{x}}$