Worksheet 6.

In this worksheet, we will explore solving equations algebraically. In our previous worksheets ($WS-2$ & $W-3$), we discussed different methods of factoring and simplifying algebraic expressions. This is more like a continuation of what we learned in those worksheets as simplifying and factoring are required when solving equations. Here we focus on solving polynomial equations (quadratic, cubic, etc.), equations involving rational functions and fractions, and radical terms (square roots, cube roots, etc.).

Solving equations often requires factoring and simplifying. The following identities are often use to simplify and factor 2^{nd} and 3^{rd} degree polynomial equations:

Identities. Let a and b be real numbers, variables, or algebraic expressions. Then

- $a^2 - b^2 = (a-b)(a+b)$
- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
- $a^3 + b^3 = (a+b)(a^2-ab+b^2)$
- $(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2$
- $(a-b)^2 = (a-b)(a-b) = a^2-2ab+b^2$
- $(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2) = a^3+3a^2b+3ab^2+b^3$
- $(a-b)^3 = (a-b)(a-b)^2 = (a-b)(a^2-2ab+b^2) = a^3-3a^2b+3ab^2-b^3$

We can often solve an equation by putting it into standard form and then using the *Zero Product Property*:

Standard Form of an equation. The standard form of an equation is:

$some\ (algebraic)\ expression = 0$

Zero Product Property. If the product of any number of expressions is zero, at least one of them must be zero. That is
If $a_1 \cdot a_2 \cdot a_3 \ldots a_n = 0$, then $a_1 = 0,\ a_2 = 0, a_3 = 0, \ldots\ a_n = 0$

Examples:

1. If $(x-3)(x+1) = 0$, then
   $(x-3) = 0$ or $(x+1) = 0$. That is $x = 3$ or $x = -1$.
2. If $3x(x-3)(x+1) = 0$, then
   $3x = 0$, $(x-3) = 0$, or $(x+1) = 0$. That is $x = 0$, $x = 3$ or $x = -1$.

Solving Polynomial Equations.
When solving polynomial equations, first write the equation in standard form and then factor completely into a product of linear factors and irreducible quadratic factors.

Examples: Solve the following equations.

1. $4x^3 + 12x^2 - 40x = 0$
   Solution:

   $4x(x^2 + 3x - 10) = 4x(x-2)(x+5) = 0$
   $4x = 0, \quad x - 2 = 0, \quad x + 5 = 0 \qquad \Rightarrow \qquad x = 0, \ x = 2, \ x = -5$

2. $x^5 - 16x = 0$
   Solution:
   $x(x^4 - 16) = x\Big((x^2)^2 - 4^2\Big) = x(x^2-4)(x^2+4) = 0 = x(x-2)(x+2)(x^2+4) = 0$
   Note that $x^2 + 4$ is an irreducibale quadratic factor. Also, $x^2 + 4 > 0$ for all x.
   $x = 0, \quad x - 2 = 0, \quad x + 2 = 0 \qquad \Rightarrow \qquad x = 0, \ x = 2, \ x = -2$

Remark:
It is important to have the equation written in the standard form and then factored in order to use the zero product property.

Examples: Solve for x.

1. $(x-3)(x+1) = 12.$
   Solution

   Even though the equation is already factored, we cannot use the zero product property as the right hand side is not zero. We first need to write the equation in standard form and then factor again.

$$
\begin{aligned}
(x-3)(x+1) - 12 &= 0\\
x^2 - 2x - 3 - 12 &= 0\\
x^2 - 2x - 15 &= 0\\
(x-5)(x+3) &= 0
\end{aligned}
$$
$x - 5 = 0, \quad x + 3 = 0 \qquad \Rightarrow \qquad x = 5, \quad x = -3.$

2. $2x^3 + x^2 - 10x = 5$
   Solution

$$
\begin{aligned}
2x^3 + x^2 - 10x &= 5\\
2x^3 - 10x + x^2 - 5 &= 0 \qquad \textit{rearrange the terms.}\\
2x(x^2 - 5) - (x^2 - 5) &= 0 \qquad \textit{factor by grouping terms.}\\
(2x-1)(x^2-5) &= 0\\
(2x-1)(x-\sqrt{5})(x+\sqrt{5}) &= 0
\end{aligned}
$$
$2x - 1 = 0, \quad x - \sqrt{5} = 0, \quad x + \sqrt{5} = 0, \qquad \Rightarrow \qquad x = 1/2, \quad x = \sqrt{5}, \quad x = -\sqrt{5}.$

When solving a quadratic equations, we may need to rely on the quadratic formula when the solutions are not rational numbers.

The Quadratic Formula. If a, b, $\&c$ are real numbers, then the solutions of the quadratic equation, $ax^2 + bx + c = 0$ can be written as:
If $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Examples: Solve for x.

1. $3x^5 + 6x^4 - 3x^3 = 0$
   Solution:

   $3x^3(x^2 + 2x - 2) = 0$
   Then $3x^3 = 0 \quad \Rightarrow \quad$ or
   $x^2 + 2x - 2 = 0 \quad \Rightarrow \quad x = \dfrac{-2 \pm \sqrt{4+4}}{2} = \dfrac{-2 \pm \sqrt{8}}{2} = \dfrac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$
   The solutions are:
   $x = 0, \quad x = -1 + \sqrt{2} \quad x = -1 - \sqrt{2}$

Try these: Solve the following equations.

1. $(x+1)(x-5) = 7$
2. $x(x^2 - x + 2) = 2$
3. $(x+1)^4 - 81 = 0$

Link to Calculus: Find critical numbers. Please note that in order to find critical numbers, we need to solve the equation $f'(x) = 0$.

1. $f(x) = x^3 + 6x^2 - 15x$
2. $f(t) = t^4 + t^3 + t^2 + 1$
3. $g(x) = 5 + 54x - 2x^3$
4. $f(x) = (x+1)^5 - 5x - 2$

Solving Equations Involving Rational Expressions.
When solving equations fractions, we often combine the fractions into a single fraction by taking the common denominator. Also, it is important to find whether there are any values such that the given algebraic expression is undefined.

Examples: Solve the following equations.

1. $\dfrac{1}{x-3} + \dfrac{1}{x+3} = \dfrac{5}{x^2 - 9}$
   Solution:

   Note that the above equation is undefined when $x = \pm 3$.
   $\dfrac{(x+3) + (x-3)}{(x-3)(x+3)} = \dfrac{5}{x^2-9}$
   $\dfrac{2x}{x^2-9} = \dfrac{5}{x^2-9} \quad \Rightarrow \quad \dfrac{2x}{x^2-9} - \dfrac{5}{x^2-9} = 0 \quad \Rightarrow \quad \dfrac{2x-5}{x^2-9} = 0 \quad \Rightarrow \quad 2x - 5 = 0 \quad \Rightarrow \quad x = \dfrac{5}{2}$

2. $2x - \dfrac{3x}{x+1} = \dfrac{1}{x+1}$
   Solution:

   Note that the above equation is undefined when $x = 1$. We can either combine terms taking the common denominator or simply multiply both sides by $(x+1)$.
   $2x(x+1) - 3x = 1$
   $2x^2 + 2x - 3x - 1 = 0$
   $2x^2 - x - 1 = 0$
   $(2x+1)(x-1) = 0 \quad \Rightarrow \quad 2x+1 = 0, \quad x - 1 = 0 \quad \Rightarrow \quad x = -1/2, \quad x = 1$

3. $\dfrac{1}{x+3} + \dfrac{1}{x+2} = 1$
   Solution:

   This equation is undefined when $x = 2$ and when $x = 3$.
   $$\frac{(x+2)+(x+3)}{(x+2)(x+3)} = 1 \quad \Rightarrow \quad \frac{2x+5}{(x+2)(x+3)} = 1$$
   $$\Rightarrow \quad 2x+5 = (x+2)(x+3) \quad \Rightarrow \quad 2x+5 = x^2+5x+6 \quad \Rightarrow \quad x^2+3x+1 = 0$$
   $$\Rightarrow \quad x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2} \quad \Rightarrow \quad x = \frac{-3+\sqrt{5}}{2}, \quad x = \frac{-3-\sqrt{5}}{2}$$

Try these: Solve the following equations.

1. $\dfrac{x^2}{x+1} - x = 4$
2. $\dfrac{1}{x(x+1)} + \dfrac{3}{x+1} = 2$
3. $\dfrac{3}{x+2} + \dfrac{x}{x-5} = \dfrac{7x}{x^2-3x-10}$

Link to Calculus: Find critical numbers. Please note that in order to find critical numbers, we need to solve the equation $f'(x) = 0$.

1. $f(x) = \dfrac{x-11}{x^2-x+1}$
2. $f(x) = \dfrac{x}{x^2+1}$
3. $g(x) = x^2 - x - \ln x$
4. $f(x) = \dfrac{x^2}{x-1}$

Solving Equations Involving Radical Expressions.
Solving equations containing radical terms can often be challenging. The following steps are helpful when solving a radical equation:

1. If there is only one radical expression involves a variable, then isolate that term.
2. If more than one radical expression involves the variable, then isolate one of them.
3. Raise both sides of the equation to the index of the radical.
4. Repeat steps 2 and 3 if it is still a radical equation.
5. Solve the resulting equation.
6. Check the answer in the original equation as some solutions may have been introduced that do not make the original equation true. This often happens by raising both sides of an equation to a power.

Remark:
It is important to understand when solving radical equations, we may have to follow other techniques such as taking common denominators and rewiring the expression prior to follow above steps depending on how the equation is written.

Examples: Solve for x.

1. $\sqrt{3x^2 - 13x - 1} - 3 = 0$.
Solution
$\sqrt{3x^2 - 13x - 1} = 3$
$3x^2 - 13x - 1 = 9 \quad \Rightarrow \quad 3x^2 - 13x - 10 = 0 \quad \Rightarrow \quad (3x + 2)(x - 5) = 0$
$\Rightarrow 3x + 2 = 0, \quad x - 5 = 0, \quad \Rightarrow \quad x = -2/3, \quad x = 5$
Next, we check the results:
When $x = -2/3$, $\quad \sqrt{3x^2 - 13x - 1} = \sqrt{3(-2/3)^2 - 13(-2/3) - 1} = \sqrt{4/3 + 26/3 - 1} = \sqrt{9} = 3$
When $x = 5$, $\quad \sqrt{3x^2 - 13x - 1} = \sqrt{3(5)^2 - 13(5) - 1} = \sqrt{75 - 65 - 1} = \sqrt{9} = 3$
Solutions: $x = -2/3, \quad x = 5$

2. $\sqrt{3x - 5} + \sqrt{x - 1} = 2$.
Solution
$\sqrt{3x - 5} = 2 - \sqrt{x - 1}$
$\left(\sqrt{3x - 5}\right)^2 = \left(2 - \sqrt{x - 1}\right)^2 = 4 - 4\sqrt{x - 1} + (x - 1)$
$3x - 5 = x + 3 - 4\sqrt{x - 1} \quad \Rightarrow \quad 4\sqrt{x - 1} = 8 - 2x \quad \Rightarrow \quad \left(4\sqrt{x - 1}\right)^2 = (8 - 2x)^2$
$\Rightarrow \quad 16(x - 1) = 64 - 32x + 4x^2 \quad \Rightarrow \quad 4x^2 - 48x + 80 = 0 \quad \Rightarrow \quad x^2 - 12x + 20 = 0$
$\Rightarrow \quad (x - 10)(x - 2) = 0 \quad \Rightarrow \quad x - 10 = 0, \quad x - 2 = 0 \quad \Rightarrow \quad x = 10, \quad x = 2.$
Next, we check the results:
When $x = 10$, $\quad \sqrt{3x - 5} + \sqrt{x - 1} = \sqrt{3(10) - 5} + \sqrt{10 - 1} = 5 + 3 = 8 \quad \Rightarrow$ Not a solution.
When $x = 2$, $\quad \sqrt{3x - 5} + \sqrt{x - 1} = \sqrt{3(2) - 5} + \sqrt{2 - 1} = 1 + 1 = 2$
Solution: $x = 2$

Try these: Solve the following equations.

1. $\sqrt{x - 10} - 4 = 0$
2. $x + \sqrt{31 - 9x} = 5$
3. $2\sqrt{x + 1} - \sqrt{2x + 3} = 1$

Link to Calculus: Find critical numbers. Please note that in order to find critical numbers, we need to solve the equation $f'(x) = 0$.

1. $f(x) = 2\sqrt{x + 1} - 3x$
2. $f(x) = x\sqrt{x^2 + 1} - 2x$